Numerical Simulation of Emden-Fowler Type Equations Using Variational Iteration Algorithm

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The main objective of this article is to present a reliable algorithm to determine exact and approximate solutions of the generalized Emden-Fowler type equations. The algorithm mainly is based on He's variational iteration method (VIM) with an alternative framework designed to overcome the difficulty of the regular singular point at $x = 0$. In this method, general Lagrange multipliers are introduced to construct a correction for the problem. The multipliers in the functional can be identified optimally via the variational theory. The results reveal that the proposed method is very effective and can be applied for other nonlinear problems.

Key words: Emden-Fowler Equation; Variational Iteration Method; Lagrange Multipliers; Correction Functional; Frobenius Method; Singular Point.

1. Introduction

The variational iteration method (VIM) was first proposed by Ji-Huan He in 1998 [1, 2]. It was systematically studied in 1999 [3] and later [4], was successfully applied to autonomous ordinary differential equations [5], boundary value problems [6], calculus of variations [7], nonlinear wave equations [8 – 11], nonlinear thermoelectricity [12], nonlinear heat transfer equations [13], astrophysics [14], biological problems [15, 16], integro-differential equations [17], nonlinear differential equations of fractional order [18], nonlinear fluid mechanics [19], circuit theory [20], and other fields [21 – 30].

By using VIM, we present in this paper a reliable algorithm to determine exact and approximate solutions of Emden-Fowler type equations of the form

$$u'' + \frac{2}{x}u' + af(x)g(u) = 0, \quad u(0) = u_0, \quad u'(0) = 0, \tag{1}$$

where $f(x)$ and $g(u)$ are some given functions of x and u, respectively. For $f(x) = 1$ and $g(u) = u^n$, (1) is the standard Lane-Emden equation which was used to model the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules [31] and subject to the classical laws of thermodynamics. For other special forms of $g(u)$, the well-known Lane-Emden equation was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behaviour of spherical cloud of gas, and isothermal gas spheres [31, 32]. A summary of the historical developments of the Emden-Fowler equation may be found in [33, 34], where an excellent bibliography on its applications is given in [34].

The proposed algorithm will then be used to investigate a generalization of (1) of the form

$$u'' + \frac{p}{x}u' + af(x)g(u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0, \quad p > 1. \tag{2}$$

We also observed that $x = 0$ is a regular singular point of (1) and (2) (see [35]). The main goal of this work is directed towards numerical solutions to find exact and approximate solutions of Emden-Fowler type equations by VIM.

2. He's Variational Iteration Method

To illustrate the basic concepts of the variational iteration method, we consider the following general nonlinear system:

$$Lu(x) + Nu(x) = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to the VIM, we can construct a correction functional as follows:

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda\,(Lu_k(\xi) + N\breve{u}_k(\xi) - g(\xi))\mathrm{d}\xi, \tag{4}$$

where λ is a general multiplier, which can be identified optimally via the variational theory [4], the subscript k denotes the kth approximation, and $\breve{u}_k$ is considered as a restricted variation [1–3], i. e. $\delta\breve{u}_k = 0$.

3. He's Variational Iteration Method for Equation (2)

In the illustrative examples which follow, we shall again seek a solution valid in some interval $0 < x < R$ (where $R > 0$). To solve (2) by means of variational iteration method, a correction functional can be written down as follows:

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda \Big\{ u_k{}''(\xi)(\xi) + \frac{p}{\xi} u_{kk}{}'(\xi) + af\xi g(\tilde{u}_k(\xi)) \Big\} \mathrm{d}\xi, \tag{5}$$

where $\tilde{u}_k$ is considered as a restricted variation. Making the correction functional and noticing that $\delta\tilde{u}_k = 0$,

$$\delta u_{k+1}(x) = \delta u_k(x) + \delta \int_0^x \lambda \Big\{ u_k{}''(\xi) + \frac{p}{\xi} u_k{}'(\xi) + af(\xi) g(\tilde{u}_k(\xi) \Big\} \mathrm{d}\xi, \tag{6}$$

then (6) yields the following stationary conditions:

$$1 + \frac{p}{x}\lambda(x) - \frac{\mathrm{d}\lambda(\xi)}{\mathrm{d}\xi}\Big|_{\xi=x} = 0, \tag{7}$$

$$\lambda(\xi)\big|_{\xi=x} = 0, \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{p}{\xi}\lambda(\xi) - \frac{\mathrm{d}\lambda(\xi)}{\mathrm{d}\xi} \right) = 0. \tag{9}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda(\xi) = \frac{\xi}{p-1}\left[\left(\frac{\xi}{x} \right)^{p-1} - 1 \right]. \tag{10}$$

As a result, we obtain the following iteration formula:

$$u_{k+1}(x) = u_k(x) + \int_0^x \frac{\xi}{p-1}\left[\left(\frac{\xi}{x} \right)^{p-1} - 1 \right] \cdot \Big\{ u_k{}'' + \frac{p}{\xi} u_k{}' + af(\xi) g(u_k) \Big\} \mathrm{d}\xi. \tag{11}$$

We start with an initial approximation $u_0(x) = u(0) + xu'(0) = \alpha$ given by (2) and use the iteration formula (11), then we obtain $u_1(x)$, $u_2(x)$, $\ldots$.

3.1. Emden-Fowler Equation with $f(x) = x^m$, $g(u) = u^n$, and $p = 2$

In this section we will discuss an Emden-Folwer equation of the form

$$u'' + \frac{2}{x}u' + ax^m u^n = 0, \tag{12}$$

which has been the object of several studies. The most interesting initial conditions are the following [21, 27]:

$$u(0) = 1, \quad u'(0) = 0. \tag{13}$$

It is important to note that for $m = 0$, (12) is the well-known Lane-Emden equation of index n. It was physically shown that interesting values of n lie in the interval $[0, 5]$. In addition, exact solutions exist only for $n = 0$, 1, and 5. For other values of n, series solutions are obtainable. The general solution (12) is constructed for all possible values of m, $n \geq 0$. Notice that (12) is linear for $n = 0$ and 1, and nonlinear otherwise.

As stated before, we can use any selective function for $u_0(x)$; preferably we use the initial condition (13), i. e.

$$u_0(x) = u(0) + xxu'(0) = 1. \tag{14}$$

For $m = 0$ and $n = 0$ (12) reduces to the equation of the form

$$u'' + \frac{2}{x}u' + a = 0. \tag{15}$$

By the above iteration formula (11), we can obtain

$$u_1(x) = 1 - \frac{1}{3!}ax^2 \tag{16}$$

and, in general, $u_k(x) = 1 - \frac{1}{3!}ax^2$ for $k \geq 2$, which can be verified through substitution.

For $m = 0$ and $n = 1$ (12) reduces to the equation of the form

$$u'' + \frac{2}{x}u' + au = 0. \tag{17}$$

Substituting (14) into (11) and after some simplifications, we obtain the following successive approximations:

$$\begin{aligned}
u_1(x) &= 1 - \frac{1}{3!}ax^2,\\
u_2(x) &= 1 - \frac{1}{3!}ax^2 + \frac{1}{5!}a^2x^4,\\
u_3(x) &= 1 - \frac{1}{3!}ax^2 + \frac{1}{5!}a^2x^4 - \frac{1}{7!}a^3x^6,\\
u_4(x) &= 1 - \frac{1}{3!}ax^2 + \frac{1}{5!}a^2x^4 - \frac{1}{7!}a^3x^6 + \frac{1}{9!}a^4x^8,
\end{aligned} \tag{18}$$

and the rest of the components of the iteration formula (11) are obtained using the Maple Package. The solution of $u(x)$ in closed form is given by

$$u(x) = \frac{\sin(\sqrt{a}x)}{\sqrt{a}x}. \tag{19}$$

A very good agreement between the results of the variational iteration method and the exact solution were observed, which confirms the validity of the He's variational iteration method.

Now, we consider another example letting be $m = 0$ and $n = 5$. For this case (12) reduces to a equation of the form

$$u'' + \frac{2}{x}u' + au^5 = 0. \tag{20}$$

Using (14) into (11) after some simplifications, we obtain:

$$\begin{aligned}
u_1(x) &= 1 - \frac{1}{6}ax^2,\\
u_2(x) &= 1 - \frac{1}{6}ax^2 + \frac{1}{24}a^2x^4 + o(x^5),\\
u_3(x) &= 1 - \frac{1}{6}ax^2 + \frac{1}{24}a^2x^4 - \frac{5}{432}a^3x^6 + o(x^7),\\
u_4(x) &= 1 - \frac{1}{6}ax^2 + \frac{1}{24}a^2x^4 - \frac{5}{432}a^3x^6\\
&\quad + \frac{35}{10368}a^4x^8 + o(x^9),
\end{aligned} \tag{21}$$

and so on. The closed form solution of $u(x)$ is given by

$$u(x) = \left(1 + \frac{1}{3}ax^2\right)^{-1/2}. \tag{22}$$

We consider the example $m = 2$ and $n = 2$. For this case (12) reduces to an equation of the form

$$u'' + \frac{2}{x}u' + ax^2u^2 = 0. \tag{23}$$

Using the selection (14) into (11) after some simplifications, we obtain the following successive approximations:

$$\begin{aligned}
u_1(x) &= 1 - \frac{1}{20}ax^4,\\
u_2(x) &= 1 - \frac{1}{20}ax^4 + \frac{1}{720}a^2x^8 + o(x^9),\\
u_3(x) &= 1 - \frac{1}{20}ax^4 + \frac{1}{720}a^2x^8 - \frac{19}{561600}a^3x^{12}\\
&\quad + o(x^{13}),\\
u_4(x) &= 1 - \frac{1}{20}ax^4 + \frac{1}{720}a^2x^8 - \frac{19}{561600}a^3x^{12}\\
&\quad + \frac{29}{38188800}a^4x^{16} + o(x^{17}),\\
&\ \vdots
\end{aligned} \tag{24}$$

which agrees exactly with the solution in [36].

Generally, for $n = 0$ we obtain the exact solution

$$\begin{aligned}
&u(x) = 1 - \frac{a}{(m+3)(m+2)}x^{m+2},\\
&m \neq -3, \ \text{and} \ m \neq -2,
\end{aligned} \tag{25}$$

for the equation

$$u'' + \frac{2}{x}u' + ax^m = 0. \tag{26}$$

3.2. *Emden-Fowler equation with $f(x) = x^m \ln x$, $g(u) = u^n$, and $p = 2$*

Consider the nonlinear Emden-Fowler type equation

$$u'' + \frac{2}{x}u' + ax^m \ln x u^n = 0, \tag{27}$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 0. \tag{28}$$

As stated before, we can use any selective function for $u_0(x)$; preferably we use the initial condition (13), i. e.

$$u_0(x) = u(0) + xu'(0) = 1. \tag{29}$$

By the above iteration formula (11), we can obtain

$$
\begin{aligned}
&u_1(x) = u_2(x) = \ldots = u(x) = 1 + \frac{3}{4}ax - \frac{1}{2}ax\ln x,\\
&u_1(x) = u_2(x) = \ldots = u(x) = 1 + \frac{5}{36}ax^2 - \frac{1}{6}ax^2\ln x,\\
&u_1(x) = u_2(x) = \ldots = u(x)\\
&\qquad = 1 + \frac{7}{144}ax^3 - \frac{1}{12}ax^3\ln x,\\
&u_1(x) = u_2(x) = \ldots = u(x)\\
&\qquad = 1 + \frac{9}{400}ax^4 - \frac{1}{20}ax^4\ln x,\\
&u_1(x) = u_2(x) = \ldots = u(x)\\
&\qquad = 1 + \frac{11}{900}ax^5 - \frac{11}{30}ax^5\ln x,
\end{aligned}
\tag{30}
$$

which are the exact solutions of the initial value problem (27)–(28) for $n = 0$ and $m = -1$, 0, 1, 2, 3, respectively.

A very good agreement between the results of the variational iteration method and the exact solution were observed, which confirms the validity of the He's variational iteration method.

For $m = 0$ and $n = 2$ (27) reduces to the equation of the form

$$u'' + \frac{2}{x}u' + a\ln x u^2 = 0. \tag{31}$$

Proceeding as before, we can start with the initial approximation (29). Using the variational iteration algorithm we obtain

$$
\begin{aligned}
u_1(x) &= 1 + \Big(\frac{5}{36} - \frac{1}{6}\ln x\Big)ax^2,\\
u_2(x) &= 1 + \Big(\frac{5}{36} - \frac{1}{6}\ln x\Big)ax^2\\
&\quad + \Big(\frac{17}{1500} - \frac{1}{6}\ln x + \frac{1}{60}\ln^2 x\Big)a^2x^4 + o(x^5),\\
u_3(x) &= 1 + \Big(\frac{5}{36} - \frac{1}{6}\ln x\Big)ax^2\\
&\quad + \Big(\frac{17}{1500} - \frac{1}{6}\ln x + \frac{1}{60}\ln^2 x\Big)a^2x^4\\
&\quad + \Big(\frac{1238791}{1555848000} - \frac{351341}{111132000}\ln x\Big)a^3x^6\\
&\quad + \Big(\frac{6079}{1587600}\ln^2 x - \frac{11}{7560}\ln^3 x\Big)a^3x^6 + o(x^5),\\
&\ \vdots
\end{aligned}
\tag{32}
$$

which agrees exactly with the solution in [36].

Now, we consider another example with $m = 2$ and $n = 3$. For this case (27) reduces to an equation of the form

$$u'' + \frac{2}{x}u' + ax^2\ln x u^3 = 0. \tag{33}$$

By the iteration formula (11) with the initial approximation (29), we can obtain the first three components as follows:

$$
\begin{aligned}
u_1(x) &= 1 + \Big(\frac{9}{400} - \frac{1}{20}\ln x\Big)ax^4,\\
u_2(x) &= 1 + \Big(\frac{9}{400} - \frac{1}{20}\ln x\Big)ax^4\\
&\quad + \Big(\frac{1231}{3110400} - \frac{83}{43200}\ln x\Big)a^2x^8\\
&\quad + \frac{1}{480}a^2x^8\ln^2 x + o(x^9),\\
u_3(x) &= 1 + \Big(\frac{9}{400} - \frac{1}{20}\ln x\Big)ax^4\\
&\quad + \Big(\frac{1231}{3110400} - \frac{83}{43200}\ln x + \frac{1}{480}\ln^2 x\Big)a^2x^8\\
&\quad + \Big(\frac{148430159}{213220672256000} + \frac{35801}{292032000}\ln^2 x\\
&\quad - \frac{36387979}{683354880000}\ln x - \frac{11}{124800}\ln^3 x\Big)a^3x^{12}\\
&\quad + o(x^{13}),\\
&\ \vdots
\end{aligned}
\tag{34}
$$

and the rest of the components of iteration formula (11) are obtained using the Maple Package. These components agree exactly with the components of the approximate solution in [36].

4. Conclusion

In this paper, we presented a new application of the He's variational iteration method to exact and approximate solution of Emden-Fowler type equations. This method can deal with highly nonlinear differential equations with no need for small parameter or linearization. All the examples show that the results of the method are in excellent agreement with those of the adomian decomposition method. The solution procedure is very simple by means of variational iteration theory, and few iterations lead to high accurate solutions in a restricted region of convergence.

This paper contains a new application of He's variational iteration method to obtain infinite series representation of solutions at singular point $x = 0$ of a Emden-Fowler equation. Also, we point out that the present technique can be extended to higher-order differential equations having singular points. In particularly, the present method can be used as alternative method to classical Frobenius method [35] to obtain infinite series solutions at the regular singular points of linear differential equations.

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